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CORRECTION TO "HARISH-CHANDRA'S PLANCHEREL THEOREM FOR $\mathfrak{p}$-ADIC GROUPS"

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K. Hiraga [1] has informed the author that the proof of Theorem 5 given in [2] is incorrect, and he has also supplied a correct proof for this result. In what follows the author will present, with insignificant modifications, the example and the argument of Hiraga's letter.

First, as Hiraga points out, the numbers $C(\overline{n})$ of [2, 3.28] are not in general finite, as the following example makes clear. We employ the notation of [2]. Let π denote a prime element of the p-field Ω. For the moment take $\mathbf{G} = \mathrm{GL}_3$, $\mathbf{P}$ the upper triangular subgroup, and $\overline{\mathbf{N}}$ the lower triangular unipotent subgroup of $\mathbf{G}$. The set F of [2, §2], a fundamental set for the Cartan decomposition of $\mathbf{G}/\mathbf{Z}$, consists of the set of matrices

$$m = \begin{bmatrix} \pi^{-\ell_1-\ell_2} & 0 & 0 \\ 0 & \pi^{-\ell_2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where ℓ_1, ℓ_2 are non-negative integers. For the example take

$$\overline{n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \pi^{-1} & 1 \end{bmatrix}.$$

Then the number of elements $C(\overline{n})$ of the set

$$\{m \in F = S(P) \mid \overline{n} \notin m^{-1}\overline{N}_j m\},$$

is not finite, since $\overline{n} \notin m^{-1}\overline{N}_j m$ if and only if $\ell_2 \leq j$ and ℓ_1 is arbitrary. Clearly, the estimate of [2, 3.29] fails.

To create valid estimates and save the concluding argument of [2, Theorem 5], expressed in equations [2, 3.30-31], Hiraga proceeds as follows. First, he partitions F via a subtler relation than was employed in [2, mid-page 4681]. Let $\Sigma^0(P, A) = \{\alpha_1, \ldots, \alpha_l\}$, the set of simple roots. Then $m \in M_0^+/{}^0M_0 Z = F$ if and only if

$$H(m) = \sum_{i=1}^{l} j_i \lambda_{\alpha_i}, \tag{1}$$

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The author wishes to thank Hiraga for informing him of the mistake in his paper and communicating in detail a correct argument. The author wishes also to take this opportunity to apologize for publishing an incorrect proof of this important theorem.

where $j_i \geq 0$ (see [2, 3.9]). For $m \in F$ define

$$\iota(m) = \min\{i \mid j_i = \max\{j_1, \dots, j_l\}\} \tag{2}$$

and

$$F_\iota = \{m \in F \mid \iota(m) = \iota\}.$$

Let $\overline{P}_0 = M_0 \ltimes \overline{N}_0$ be the opposite minimal parabolic subgroup of P_0. As the argument of [2, Theorem 5] implies, it is sufficient to show that, for $f \in {}^0\mathcal{C}(G)$,

$$\sum_{m \in F_\iota} \left| \int_{m^{-1}(\overline{N}_0 \cap K_j)m} f(\overline{n}_0) \right| < \infty \tag{3}$$

for every $\iota = 1, \dots, l$. Thus it suffices to fix ι and to prove (3). Let $\overline{P} = M \ltimes \overline{N}$ be the opposite parabolic subgroup of the standard maximal parabolic subgroup corresponding to $\Sigma^0(P_0, A_0) - \{\alpha_\iota\}$ and let ${}^*\overline{N} = M \cap \overline{N}_0$. Then

$$\overline{N}_0 = {}^*\overline{N} \ltimes \overline{N}$$

and

$$\overline{N}_0 \cap K_j = ({}^*\overline{N} \cap K_j) \ltimes \overline{N}_j.$$

Since $f \in {}^0\mathcal{C}(G)$,

$$\begin{aligned} \int_{m^{-1}(\overline{N}_0 \cap K_j)m} f(\overline{n}_0)\, d\overline{n}_0 &= \int_{m^{-1}({}^*\overline{N} \cap K_j)m} \int_{m^{-1}\overline{N}_j m} f({}^*\overline{n} \cdot \overline{n})\, d\,{}^*\overline{n}\, d\overline{n} \\ &= -\int_{m^{-1}({}^*\overline{N} \cap K_j)m} \int_{\overline{N} - m^{-1}\overline{N}_j m} f({}^*\overline{n} \cdot \overline{n})\, d\,{}^*\overline{n}\, d\overline{n}. \end{aligned}$$

For any $\overline{n}_0 = {}^*\overline{n} \cdot \overline{n} \in \overline{N}_0$ we define $C({}^*\overline{n} \cdot \overline{n})$ to be the cardinality of the set

$$\begin{aligned} S({}^*\overline{n} \cdot \overline{n}) &= \{m \in F_\iota \mid {}^*\overline{n} \cdot \overline{n} \in m^{-1}({}^*\overline{N} \cap K_j)m \cdot (\overline{N} - m^{-1}\overline{N}_j m)\} \\ &= \{m \in F_\iota \mid m\, {}^*\overline{n} m^{-1} \in {}^*\overline{N} \cap K_j \text{ and } m\overline{n}m^{-1} \notin \overline{N}_j\}. \end{aligned} \tag{4}$$

Lemma 3. *$C({}^*\overline{n} \cdot \overline{n}) < \infty$ and, more precisely, $C({}^*\overline{n} \cdot \overline{n}) \prec (\sigma({}^*\overline{n} \cdot \overline{n}) + 1)^l$.*

Proof. It is enough to check that $C({}^*\overline{n} \cdot \overline{n}) \prec (\sigma(\overline{n})+1)^l$, since $\sigma(\overline{n})+1 \prec \sigma({}^*\overline{n} \cdot \overline{n})+1$. In particular, it suffices to show that

$$|\{m \in F_\iota \mid m\overline{n}m^{-1} \notin \overline{N}_j\}| \prec (\sigma(\overline{n}) + 1)^l, \tag{5}$$

since the additional condition of (4) can only make the left side of (5) smaller. We recall that the roots which occur in the Lie algebra of $\overline{\mathbf{N}}$ consist of exactly those negative A_0 roots β such that $\langle \lambda_{\alpha_\iota}, \beta \rangle < 0$. This means that $\overline{N}$ may be regarded as a product of lines, each line a copy of the gound field Ω and each line contracted by the effect of multiplication by a rational character of M_0 under the mapping $\overline{n} \mapsto m\overline{n}m^{-1}$ for any $m \in F_\iota$ (assuming, as we may, that $j_\iota > 0$ in (1)). The function $\overline{n} \mapsto \sigma(\overline{n})$ has non-negative values on $\overline{N}$, is positive outside some compact neighborhood of the identity, and is bounded on compact subsets of $\overline{N}$. More precisely, there exist positive constants $C(\iota, j)$ such that if j_ι in (1) is chosen such that

$$j_\iota > C(\iota, j)(\sigma(\overline{n}) + 1), \tag{6}$$

then $m\overline{n}m^{-1} \in \overline{N}_j$. Using [2, Lemma 2], (1), and (2), we have

$$|\{m \in F_\iota \mid j_\iota = k\}| = O(k^l) \qquad (k > 0). \tag{7}$$

Combining (6) and (7) completes the proof. □

Using Lemma 3, we can now prove (3) almost by repeating the argument of [2, 3.27, 3.30-31]:

$$\sum_{m \in F_\iota} \left| \int_{m^{-1}(\overline{N}_0 \cap K_j)m} f(\overline{n}_0)\, d\overline{n}_0 \right| \prec \int_{\overline{N}_0} \delta_{P_0}^{-1/2}(\overline{n}_0) C(\overline{n}_0)(1+\sigma(\overline{n}_0))^{-r}\, d\overline{n}_0$$
$$\prec \int_{\overline{N}_0} \delta_{P_0}^{-1/2}(\overline{n}_0)(1+\sigma(\overline{n}_0))^{-r+l}\, d\overline{n}_0 < \infty$$

for all r sufficiently large.[1]

References

1. Kaoru Hiraga, *Letter to Allan J. Silberger dated February 23, 1999.*
2. Allan J. Silberger, *Harish-Chandra's Plancherel Formula for $\mathfrak{p}$-adic Groups*, Trans. Amer. Math. Soc. **348** (1996), 4673–4686. MR **99c:**22026

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[1]Please note: The groups $\overline{N}_j$ and N_j of [2, p. 4674, the last paragraph] are not normal subgroups of K_j; each is normalized by K_j^M.